

# HANDLE DECOMPOSITIONS OF RATIONAL BALLS AND CASSON–GORDON INVARIANTS

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**ABSTRACT.** Given a rational homology sphere which bounds rational homology balls, we investigate the complexity of these balls as measured by the number of 1-handles in a handle decomposition. We use Casson–Gordon invariants to obtain lower bounds which also lead to lower bounds on the fusion number of ribbon knots. We use Levine–Tristram signatures to compute these bounds and produce explicit examples.

## 1. INTRODUCTION

Given two concordant knots it is natural to ask how complicated a concordance between them must be. An similar question can be asked about (rational) homology cobordant 3-manifolds and cobordisms between them. Very little is known about these two simple and natural questions.

In the context of knot concordance a natural notion of complexity already considered by several authors [15, 12] is that of the *fusion number* of a ribbon knot. The analogous notion for integral or rational homology spheres bounding an integral or rational homology ball is the minimum number of 1-handles needed to construct an integral or rational homology ball bounding the given 3-manifolds. These numerical invariants encode deep 4-dimensional information of knots and 3-manifolds and are extremely hard to compute. Motivation in this direction comes also from analogous questions which are purely 4-dimensional. One of the oldest open problems in smooth 4-manifold topology asks if it is true that every smooth simply-connected 4-manifold has a handle decomposition with no 1-handles.

In this paper we investigate the complexity of rational balls (as measured by the number of handles in their handle decompositions) bounded by a given rational homology sphere. More precisely we consider the following question.

**Question 1.1.** Let  $Y$  be a rational homology sphere which bounds a rational ball. What is the minimal number of 1-handles needed to realise a rational ball bounded by  $Y$ ? What if we restrict to those rational balls constructed only with 1- and 2-handles?

We provide lower bounds on these numbers using Casson–Gordon invariants. To each rational homology sphere  $Y$  and a character  $\varphi : H_1(Y) \rightarrow \mathbb{C}^*$  these associate a rational number  $\sigma(Y, \varphi)$ . We relate this number to handle decompositions of rational balls bounding  $Y$  via the following statement.

**Theorem 1.2.** *Let  $Y$  be a 3-manifold that bounds a rational homology ball  $W$ , and let  $\varphi : H_1(Y) \rightarrow \mathbb{C}^*$  be a nontrivial character that factors through  $H_1(W)$ . Every handle decomposition of  $W$  contains at least  $|\sigma(Y, \varphi)| - 1$  odd-index handles.*

In order to use the above theorem one needs to understand for which  $Y$  there exist characters which factor through *any* rational ball bounded by  $Y$ . One example of this situation is described in the following corollary.

**Corollary 1.3.** *Let  $Y$  be a 3-manifold with  $H_1(Y)$  cyclic of order  $m^2$  and  $\varphi : H_1(Y) \rightarrow \mathbb{Z}_k$  be a nontrivial character with  $k|m$ . If  $W$  is a rational homology ball with  $\partial W = Y$ , then every handle decomposition of  $W$  contains at least  $|\sigma(Y, \varphi)| - 1$  odd-index handles.*

In order to produce specific examples we need an efficient way to compute Casson–Gordon invariants. Using work of Cimasoni and Florens and focusing on 3-manifolds obtained via Dehn surgery on knots we reduce our problem to a computation of Levine–Tristram signatures.

**Proposition 1.4.** *If  $Y = S_{m^2}^3(K)$  bounds a rational homology ball  $W$  with one 1-handle and no 3-handles, then  $|1 - \sigma_K(e^{2a\pi i/m}) - 2a(m - a)| \leq 1$  for every  $1 \leq a < m$  such that  $(a, m) = 1$ .*

Note that  $S_{m^2}^3(K)$  bounds a rational homology ball with one 1-handle and no 3-handles if and only if it can be obtained via Dehn surgery on a knot in  $S^1 \times S^2$  and therefore we obtain an obstruction for this last property as well.

The examples obtained include the following:

- the 3-manifold  $Y = L(25, 21) \# L(4, 3)$  bounds no rational homology ball with a single 1-handle and it bounds one built with two 1-handles and two 2-handles;
- the 3-manifold  $S_{400}^3(T_{4,25;2,201})$  (here  $T_{4,25;2,201}$  is the  $(2, 201)$ -cable of the torus knot  $T_{4,25}$ ) bounds no rational homology ball with a single 1-handle and it bounds one built with three 1-handles and three 2-handles.

Finally we use this machinery to provide lower bounds on the *fusion number* of ribbon knots, i.e. the minimal number of 1-handles used to construct a ribbon disk. This is done by considering the branched double cover of a ribbon knot and then implementing the available obstructions.

**Organisation of the paper.** In Section 2 we develop the lower bounds based on Casson–Gordon signatures and, in specific situations, we relate these invariants to Tristram–Levine signatures. In Section 3 we give some examples.

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## 2. CASSON–GORDON SIGNATURES AND HANDLE DECOMPOSITIONS

We briefly recall the definition of Casson–Gordon signature invariants [6] and set up some notation. In what follows  $H_*(X)$  will denote the homology of  $X$  with integer coefficients and  $C_m$  the cyclic group of  $m$  elements.

Let  $(Y^3, \varphi)$  be a rational homology 3-sphere with a multiplicative character  $\varphi : H_1(Y) \rightarrow \mathbb{C}^*$ . The bordism group  $\Omega_3(K(C_m, 1))$  is finite for each  $m \in \mathbb{Z}_+$ , and therefore  $r$  copies of  $(Y^3, \varphi)$  bound a pair  $(X^4, \psi)$ , where  $\psi : H_1(X) \rightarrow C_m \subset \mathbb{C}^*$  restricts to  $\varphi$  on each of the  $r$  boundary components. Let  $\tilde{X}$  denote the  $m$ -fold cover of  $X$  corresponding to  $\psi$  with group of deck transformations isomorphic to  $C_m$ . This action induces a  $\mathbb{Z}[C_m]$ -module structure on  $H_2(\tilde{X})$ . Recall that given  $\zeta_m$ , a primitive root of unity of order  $m$ , the cyclotomic field  $\mathbb{Q}(\zeta_m)$  is a natural  $\mathbb{Z}[C_m]$ -module and we can define the twisted homology group

$$H_2^\psi(X; \mathbb{Q}(\zeta_m)) := H_2(\tilde{X}; \mathbb{Q}) \otimes_{\mathbb{Z}[C_m]} \mathbb{Q}(\zeta_m).$$

This group admits a  $\mathbb{C}$ -valued Hermitian intersection form whose signature will be denoted by  $\sigma^\psi(X)$ . The signature of the standard intersection pairing on  $H_2(X)$  will be denoted by  $\sigma(X)$ . The *Casson–Gordon signature invariant* of the pair  $(Y, \varphi)$  is given by the difference:

$$(1) \quad \sigma(Y, \varphi) := \frac{1}{r} (\sigma^\psi(X) - \sigma(X)).$$

Our main result provides a bound on the complexity of rational homology balls in terms of their handle decompositions. The bound is established making use of the Casson–Gordon signature invariant. The proof of this result is very similar in nature to the one in the original paper of Casson and Gordon [6, Theorem 1] but with a different application in mind.

**Theorem 2.1.** *Let  $Y$  be a 3-manifold that bounds a rational homology ball  $W$ , and let  $\varphi : H_1(Y) \rightarrow \mathbb{C}^*$  be a nontrivial character that factors through  $H_1(W)$ . Every handle decomposition of  $W$  contains at least  $|\sigma(Y, \varphi)| - 1$  odd-index handles.*

*Proof.* Let  $\psi : H_1(W) \rightarrow \mathbb{C}^*$  be a character that extends  $\varphi$ , namely,  $\varphi = i^* \psi$ , where  $i : Y \hookrightarrow W$  is the inclusion. We shall use the manifold  $W$  to compute  $\sigma(Y, \varphi)$  as in (1). In this case  $r = 1$  and since  $W$  is a rational homology ball,  $H_2(W)$  is torsion and hence  $\sigma(W) = 0$ . Therefore, in (1), we are only concerned with the first summand,  $\sigma^\psi(W)$ .

We denote by  $\widetilde{W}$  the covering associated to  $\psi$  and fix  $m$  to be the order of  $\psi(H_1(W)) \subset \mathbb{C}^*$ . Any cell decomposition of  $W$  induces a chain complex of the covering  $C_*(\widetilde{W})$ , which we view as generated over  $\mathbb{Z}[C_m]$  by one lift of each cell in the given decomposition of  $W$ . The module structure allows us to consider the twisted chain complex

$$C_k^\psi(W) := C_k(\widetilde{W}) \otimes_{\mathbb{Z}[C_m]} \mathbb{Q}(\zeta_m),$$

with associated homology  $H_*^\psi(W)$  and Euler characteristic  $\chi^\psi(W)$ . Note that, since  $W$  is a rational homology ball and has  $\chi(W) = 1$ , then also  $\chi^\psi(W) = 1$ . Moreover, remark that the  $k$ -th twisted Betti number  $b_k^\psi(W) := \text{rk } H_k^\psi(W)$  of  $W$  is bounded from above by the number  $n_k$  of  $k$ -cells in the decomposition of  $W$ .

The quantity  $\sigma^\psi(W)$  in formula (1) is obviously bounded by  $b_2^\psi(W)$  and since  $\psi$  is nontrivial by assumption,  $H_0^\psi(W) = 0$ . Therefore, since

$$1 = \chi^\psi(W) = -b_1^\psi(W) + b_2^\psi(W) - b_3^\psi(W),$$

we have

$$|\sigma(Y, \varphi)| = |\sigma^\psi(W)| \leq b_2^\psi(W) = 1 + b_1^\psi(W) + b_3^\psi(W) \leq 1 + n_1 + n_3.$$

as desired.  $\square$

The statement of Theorem 2.1 requires that a character defined on a 3-manifold extends over a rational homology 4-ball. In Corollary 2.2 we deal with a particular case in which the extension of the character is guaranteed and in Corollary 2.3 we give a bound on fusion number of ribbon knots.

**Corollary 2.2.** *Let  $Y$  be a 3-manifold with  $H_1(Y)$  cyclic of order  $m^2$  and  $\varphi : H_1(Y) \rightarrow C_k$  be a nontrivial character with  $k|m$ . If  $W$  is a rational homology ball with  $\partial W = Y$ , then every handle decomposition of  $W$  contains at least  $|\sigma(Y, \varphi)| - 1$  odd-index handles.*

*Proof.* Using the long exact sequence for the pair  $(W, Y)$ , it is not difficult to show that the image of  $i_* : H_1(Y) \rightarrow H_1(W)$  has order  $m$ . It follows that, whenever  $k|m$ ,  $\ker i_* \subset \ker \varphi$ , and hence  $\varphi$  factors through the image of  $i_*$ , giving  $\psi_0 : i_*(H_1(Y)) \rightarrow \mathbb{Q}/\mathbb{Z}$ , where we look at  $\mathbb{Q}/\mathbb{Z}$  as the set of roots of unity in  $\mathbb{C}^*$ . Since  $\mathbb{Q}/\mathbb{Z}$  is an injective  $\mathbb{Z}$ -module, we can extend  $\psi_0$  to  $H_1(W)$ , hence obtaining an extension  $\psi : H_1(W) \rightarrow \mathbb{Q}/\mathbb{Z} \subset \mathbb{C}^*$ .

Therefore, the assumptions of Theorem 2.1 are satisfied, and the result follows.  $\square$

We now turn to give a lower bound on the fusion number of a ribbon knot. In what follows, given a knot  $K \subset S^3$ , we will denote with  $\det K$  the determinant of  $K$  and with  $\Sigma(K)$  the double cover of  $S^3$  branched over  $K$ .

**Corollary 2.3.** *Let  $K$  be a ribbon knot with fusion number  $b$ . Then*

$$b \geq \min_H \max_\varphi |\sigma(Y, \varphi)| - 1,$$

where the minimum is taken over all subgroups  $H < H_1(\Sigma(K))$  of index  $\sqrt{|\det K|}$  and the maximum is taken over all characters  $\varphi$  whose kernel contains  $H$ .

*Proof.* Let  $D$  be a disk in the 4-ball realizing the fusion number for  $K$ . The double cover of  $B^4$  branched over this disk is a rational homology ball  $W$  bounded by  $\Sigma(K)$ , the double cover of  $S^3$  branched over  $K$ . By Theorem 2.1, from every character  $\varphi$  defined on  $H_1(\Sigma(K))$  which extends over the rational homology ball we obtain a lower bound,  $|\sigma(Y, \varphi)| - 1$ , on the number of odd-index handles in any decomposition of  $W$ .

Now, since  $D$  is a ribbon disk,  $W$  can be built with no 3-handles and with only  $b$  1-handles. It is well known that the order of  $H_1(\Sigma(K))$  is equal to  $|\det K|$  and, since there are no 3-handles in  $W$ , the order of  $H_1(W)$  equals  $\sqrt{|\det K|}$  and the inclusion  $i_* : H_1(\Sigma(K)) \rightarrow H_1(W)$  is a surjection. Notice that a character  $\varphi$  extends if and only if  $\ker i_* \subset \ker \varphi$ , which in this case can be translated into  $\varphi$  vanishing on a subgroup of  $H_1(\Sigma(K))$  of order  $\sqrt{|\det K|}$ . The result follows.  $\square$

We now focus on the special case of 3-manifolds  $Y$  obtained as surgery on a knot  $K \subset S^3$ . We shall denote such manifolds as  $S_r^3(K)$ , where  $r \in \mathbb{Q}$  is the surgery coefficient. For this class of manifolds we will give a bound on the complexity of a rational homology ball bounded by  $Y$  in terms of  $\sigma_K(\omega_m)$ , the Levine–Tristram signature of the knot  $K$  evaluated at a primitive root of unity of order  $m$ . The precise statement follows.

**Proposition 2.4.** *If  $Y = S^3_{m^2}(K)$  bounds a rational homology ball  $W$  with one 1-handle and no 3-handles, then  $|1 - \sigma_K(e^{2a\pi i/m}) - 2a(m - a)| \leq 1$  for every  $1 \leq a < m$  such that  $(a, m) = 1$ .*

*Proof.* Under these assumptions,  $\pi_1(W)$  is cyclic, any surjective character  $\varphi : H_1(Y) \rightarrow C_m$  factors through the map  $H_1(Y) \rightarrow H_1(W)$  induced by the inclusion, and there is an extension of  $\varphi$  to  $\psi : H_1(W) \rightarrow \mathbb{C}^*$  (see the proof of Theorem 2.2). We can therefore use  $(W, \psi)$  to compute the Casson–Gordon signature invariant of the pair  $(Y, \varphi)$ . Notice that in this case in formula (1) we have  $r = 1$  and  $\sigma(W) = 0$ , so  $\sigma(Y, \varphi) = \sigma^\psi(W)$ .

We proceed now to estimate  $|\sigma^\psi(W)|$ . Since there are no 3-handles,  $H_1(W) = C_m$ , the covering associated to  $\psi$  is the universal cover of  $W$  and therefore  $H_1^\psi(W) = 0$ , which implies  $b_1^\psi(W) = 0$ . Since  $H_1^\psi(W) = 0$ , we deduce, from the long exact sequence for the pair  $(W, Y)$ , that  $H_1^\psi(W, Y) = 0$ . By Poincaré duality it follows that  $H_\psi^3(W) = 0$ , and thus  $b_3^\psi(W)$  also vanishes. Finally, since  $\psi$  is non trivial, we have  $b_0^\psi(W) = 0$ . Now, since  $W$  is a rational homology ball,

$$\chi(W) = 1 = \chi^\psi(W) = b_0^\psi(W) - b_1^\psi(W) + b_2^\psi(W) - b_3^\psi(W) = b_2^\psi(W)$$

and we obtain that  $\dim H_2^\varphi(W) = 1$  and hence the signature  $\sigma^\psi(W)$  of the equivariant intersection form is bounded by 1 in absolute value, and therefore  $|\sigma(Y, \varphi)| \leq 1$ .

To finish the proof, we rewrite the Casson–Gordon signature invariant of  $Y$  in terms of the Levine–Tristram signature of the surgery knot  $K$  using the formula in Cimasoni–Florens [7, Theorem 6.7]. To this end, identify  $C_m$  with the cyclic group generated by  $\omega_m = e^{2i\pi/m} \in \mathbb{C}^*$  by sending  $1 \in C_m$  to  $e^{2i\pi/m}$  and denote by  $\sigma_K(\cdot)$  the Levine–Tristram signature of a knot. Let  $\omega_m^a$  be the image of the meridian of  $K$  under the character  $\varphi$ . The statement of the proposition then follows from the equality:

$$|\sigma(Y, \varphi)| = |\sigma_K(\omega_m^a) - 1 + \frac{2}{m^2}(m - a)am^2|. \quad \square$$

**Remark 2.5.** Notice that, when working with a rational homology sphere  $Y$  that bounds a rational homology ball  $W$  without 3-handles, one can give a lower bound on the complexity of  $W$  by looking at  $H_1(Y)$ : if  $H_1(Y)$  is generated by no fewer than  $g$  generators, every handle decomposition of  $W$  contains at least  $\lfloor g/2 \rfloor$  1-handles.

To prove this, take a handle decomposition of  $W$  with  $b$  1-handles and  $b$  2-handles, and do a dot-zero surgery along the core of each 1-handle. This presents  $Y$  as an integer surgery along a  $2b$ -component link, and correspondingly presents  $H_1(Y)$  as a quotient of  $\mathbb{Z}^{2b}$ , hence  $2b \geq g$ .

The statement of Proposition 2.4 can be extended to surgeries with rational coefficients. As shown in Figure 1, a rational surgery on a knot  $K$  can be interpreted as an integral surgery on a link  $L = K \cup U_2 \cup \dots \cup U_n$  where all the  $U_i$ 's are unknots. We will use this link  $L$  to compute the Casson–Gordon signature invariants of  $Y = S^3_{m^2/q}(K)$ . Notice that any character  $\varphi : H_1(Y) \rightarrow C_m$  can be determined from a character defined on  $H_1(S^3 \setminus L)$  sending the meridian of  $K$  to  $a \in C_m$  and extending to  $H_1(Y)$ . If in the link  $L$  we replace  $K$  with an unknot  $U_1$  and leave the same surgery coefficients, we obtain a surgery description of the lens space  $L(m^2, -q) = S^3_{m^2/q}(U)$  and a character

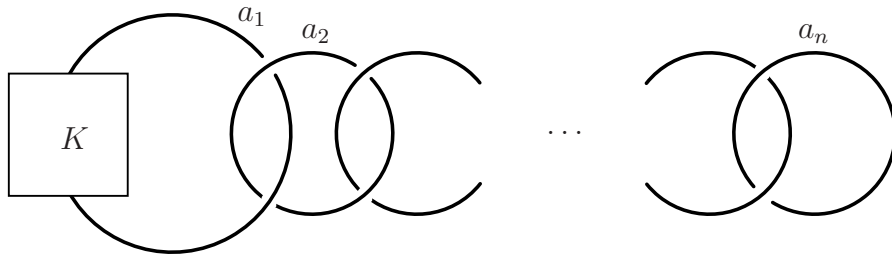


FIGURE 1. The integral surgery picture for  $p/q$ -surgery along  $K$ , where  $[a_1, \dots, a_n]^-$  is the negative continued fraction expansion of  $p/q$ .

$\chi_a : H_1(L(m^2, -q)) \rightarrow C_m$  sending the meridian of  $U_1$  to  $a$ . With all these conventions in place, we have the following statement.

**Proposition 2.6.** *If  $Y = S^3_{m^2/q}(K)$  bounds a rational homology ball  $W$  with one 1-handle and no 3-handles, then  $|\sigma_K(e^{2i\pi a/m}) + \sigma(L(m^2, -q), \chi_a)| \leq 1$  for every  $1 \leq a < m$  such that  $(a, m) = 1$ .*

**Remark 2.7.** There is an explicit formula for  $\sigma(L(m^2, -q), \chi_a)$  given by Gilmer [10, Example 3.9].

*Proof of Proposition 2.6.* The same arguments used in Proposition 2.4 allow us to conclude in this case that any surjective character  $\varphi : H_1(Y) \rightarrow C_m$  has an extension  $\psi$  to  $W$ ,  $\sigma(Y, \varphi) = \sigma^\psi(W)$  and, since  $b_2^\psi = 1$ , we have  $|\sigma(Y, \varphi)| \leq 1$ .

To finish the proof, we want to express  $\sigma(Y, \varphi)$  using [7, Theorem 6.7] applied to the surgery diagram depicted in Figure 1. We refer the reader to [7] for the pertinent definitions. The formula given by Cimasoni and Florens has one term that depends on the knot  $K$ , the colored signature of  $L$ , and all the others, which we will denote by  $T_{\varphi, \Lambda}$ , depend exclusively on the image of the meridians of  $L$  via  $\varphi$  and on the linking matrix  $\Lambda$  of the surgery presentation of  $Y$ . It follows that, with the exception of the first term in the formula, all the others remain unchanged if we substitute  $K$  with an unknot. That is, if we compute the Casson–Gordon invariant of a lens space from the chain surgery presentation with coefficients  $[a_1, \dots, a_n]$  and for the character that is defined by sending the meridian of  $U_1$  to  $e^{2i\pi a/m}$ . This Casson–Gordon invariant is precisely  $\sigma(L(m^2, -q), \chi_a)$ .

Now, notice that the link  $L$  bounds an evident  $C$ -complex in the sense of [7] given by a Seifert surface for  $K$  and a series of embedded disks, one for each unknot. The first homology of this complex coincides with the first homology of the Seifert surface for  $K$ , and the multivariable colored signature of  $L$  evaluated at any vector of roots of unity  $(\omega_1, \dots, \omega_n)$  coincides with the Levine–Tristram signature of  $K$  evaluated at  $\omega_1$ . This yields

$$\sigma(Y, \varphi) = \sigma_L(\omega) + T_{\varphi, \Lambda} = \sigma_K(\omega_1) + T_{\varphi, \Lambda}.$$

Since there is an evident contractible  $C$ -complex for the chain surgery presentation of  $L(m^2, -q)$ , it follows that  $\sigma(L(m^2, -q), \chi_a) = T_{\varphi, \Lambda}$  and, by definition of  $\varphi$ , we have  $\omega_1 = e^{2i\pi a/m}$ . The result follows.  $\square$



## 3. EXAMPLES

**Example 3.1.** As promised in the introduction, we prove that the 3-manifold  $Y = L(25, 21) \# L(4, 3)$  bounds no rational homology ball constructed with a single 1-handle and no 3-handles. However, it bounds a rational homology ball built with two 1-handles and two 2-handles.

Indeed, as shown by Moser [16],  $S_{100}^3(T_{4,25}) = L(25, 21) \# L(4, 3)$ ; we can now use Proposition 2.4 to obstruct the existence of such a ball. In fact, using the formula from [14], we see that  $\sigma_{T_{4,25}}(e^{i\pi/5}) = -15$  (note that  $e^{i\pi/5}$  is a root of the Alexander polynomial of  $T_{4,25}$ , thus explaining why the signature is odd), and therefore

$$\sigma(Y, \varphi) = |1 - \sigma_{T_{4,25}}(e^{i\pi/5}) - 2(10 - 1)| = |1 + 15 - 18| = 2.$$

Since each of  $L(25, 21)$  and  $L(4, 3)$  bounds a rational homology ball built with a single 1-handle and a single 2-handle [4], their connected sum does indeed bound a rational homology ball, built with two 1-handles and two 2-handles, namely the boundary connected sum of the two balls above.

Also, note that  $L(25, 7)$  is a lens space that bounds a rational homology ball  $W$ , and that  $L(25, 7) \# L(25, 7)$  bounds a rational homology ball built with a single 1-handle and a single 2-handle, which is therefore simpler than the boundary connected sum of two copies of  $W$ . This shows that the example above is nontrivial.

**Example 3.2.** In fact, the previous example readily generalises to the following family: whenever  $Y_{a,b} = L(a^2, -b^2) \# L(b^2, -a^2) = S_{(ab)^2}^3(T_{a^2, b^2})$  and  $\lfloor b/a \rfloor \leq 2$  bounds a rational homology ball, we will show that for the character  $\varphi$  on  $Y_{a,b}$  that maps the meridian to  $\exp(2\pi i/ab)$ ,  $|\sigma(Y_{a,b}, \varphi)| \geq 2$ , thus showing that  $Y_{a,b}$  does not bound a rational homology ball with one 1-handle and no 3-handles.

Note that if  $ab$  is odd,  $Y_{a,b}$  is the branched double cover of a ribbon knot by work of Lisca [13], and for every character  $\varphi'$  of order a *prime power*,  $|\sigma(Y_{a,b}, \varphi')| \leq 1$  [6, Theorem 2]. As a concrete example, we can choose  $a = 5$ ,  $b = 13$ ; in this case,  $Y_{a,b}$  bounds a rational homology ball, obtained as the complement of a rational cuspidal curve in  $\mathbb{CP}^2$  (see [18, 9]).

Indeed, infinitely many pairs of odd integers arise in this fashion: if we denote with  $F_n$  the  $n$ -th Fibonacci number (where we let  $F_0 = 0, F_1 = 1$ ), for each pair  $(a, b) = (F_{6k-1}, F_{6k+1})$  the 3-manifold  $Y_{a,b}$  bounds a rational homology ball [18, 9] and  $H_1(Y_{a,b})$  has odd order, since both  $F_{6k-1}$  and  $F_{6k+1}$  are odd.

We remark here that, when  $ab$  is odd,  $Y_{a,b}$  is the branched double cover of a knot  $K_{a,b}$  which is a connected sum of two 2-bridge knots; when  $(a, b) = (5, 13)$ , for instance,  $K_{5,13} = K(169/25) \# K(25/14)$ , and each of the two summands is ribbon with fusion number 1. However,  $K_{5,13}$  has fusion number 2: indeed, by Corollary 2.3 its fusion number is at least 2, and since it is the connected sum of two fusion number-1 knots, the inequality is sharp.

These examples show, once again, that the assumption that the order of  $\varphi'$  is a prime power in [6, Theorem 2] is indeed essential.

We now prove the claim above that  $|\sigma(Y_{a,b}, \varphi)| \geq 2$ . Since

$$\sigma(Y_{a,b}, \varphi) = 3 - \sigma_{T_{a^2, b^2}}(e^{2\pi i/ab}) - 2ab,$$

it suffices to show that  $\sigma_{T_{a^2,b^2}}(e^{2\pi i/ab}) \geq -2ab + 5$ . Observe that the Alexander polynomial of  $T_{a^2,b^2}$  has simple roots, and there are  $a^2b^2$ -th roots of unity that are neither  $a^2$ -th nor  $b^2$ -th roots of 1. By these two observations, we know that  $\sigma_{T_{a^2,b^2}}$  can only jump by 2 at each such root of unity, and that the value at the root of unity is the average of the neighbouring values; since the signature vanishes at 1, at  $e^{2\pi i(\frac{1}{ab}-\varepsilon)}$  it is bounded by the number of roots in the arc  $e^{2\pi it}$  with  $t \in [0, \frac{1}{ab} - \varepsilon]$ . These are easily counted to be  $ab - \lfloor b/a \rfloor \geq ab - 2$ . Since  $e^{2\pi i/ab}$  is a root of the Alexander polynomial,  $\sigma_{T_{a^2,b^2}}(e^{2\pi i/ab}) \geq -2ab + 5$ , as desired.

**Example 3.3.** One can refine the example above to produce an *irreducible* surgery that bounds no rational homology ball with one 1-handle, but does bound one with three. In fact,  $Y = S_{400}^3(T_{4,25;2,201})$  bounds a rational homology ball [3], and a quick computation with the Levine–Tristram signature using [14] yields

$$\sigma(Y, \varphi) = |1 - \sigma_{T_{4,25;2,201}}(e^{i\pi/10}) - 2 \cdot (20 - 1)| = |\sigma_{2,201}(e^{i\pi/10}) - 37| = |35 - 37| = 2,$$

where  $\varphi$  is the character that sends the meridian to  $e^{2i\pi/20}$ .

Irreducibility is proven by looking at the canonical plumbing diagram for  $Y$ ; since it is connected,  $Y$  is irreducible (see [17, 8] for details).

We can also find examples when  $H_1(Y)$  has odd order. Indeed, one can look at  $Y = S_{9 \cdot 25 \cdot 169}^3(T_{25,169;3,3 \cdot 25 \cdot 169+1})$  and the character  $\varphi$  that sends a meridian to  $e^{2\pi i/(3 \cdot 5 \cdot 13)}$ : a similar computation to the one above with the Levine–Tristram signatures yields:

$$\sigma(Y, \varphi) = |1 - \sigma_{T_{25,169;3,3 \cdot 25 \cdot 169+1}}(e^{2\pi i/(3 \cdot 5 \cdot 13)}) - 2 \cdot (3 \cdot 5 \cdot 13 - 1)| = 2,$$

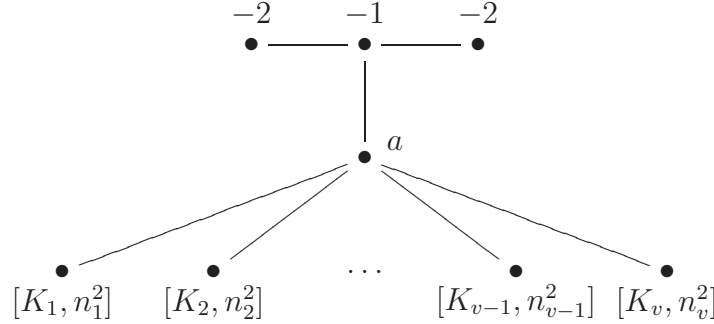
hence proving that  $Y$  does not bound a rational homology ball with no 3-handles and a single 1-handle.

We conclude with a rather lengthy example where we produce a family of irreducible 3-manifold with cyclic  $H_1$ , each of which bounds a rational homology ball built with handles of index at most 2, but such that the number of handles needed is arbitrarily large.

The structure of the argument will be the following: we fix an integer  $v$ , and we construct a 3-manifold  $Y$  by a construction that is akin to the plumbing of spheres. The construction of  $Y$  depends on the choice of  $v$  knots  $K_1, \dots, K_v \subset S^3$  and  $v + 1$  integers  $a, n_1, \dots, n_v$ , and we show that, under certain assumptions, all these manifolds have cyclic  $H_1$ . We then specialise to a certain family of knots  $K_j$  and integers  $n_j$ , coming from the example above, and we prove that the resulting  $Y$  does indeed bound a rational homology ball built out of  $2v + 1$  1-handles and  $2v - 1$  2-handles. We then compute the signature defect associated to a certain character  $\varphi$  of  $Y$  within an error of 2, and using Corollary 2.2 we show that any rational homology ball 2-handlebody needs at least  $2v - 1$  1-handles. Finally, we argue the irreducibility of  $Y$ .



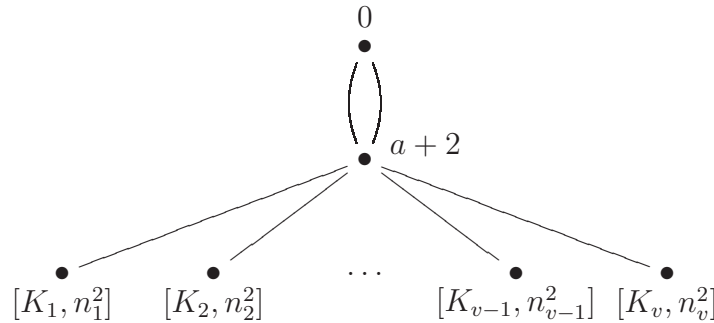
**Example 3.4.** Let us consider the following (modified) plumbing diagram, representing a 3-manifold  $Y$ :



Where each label  $[K, n]$  at the bottom signifies that, in the corresponding surgery picture for  $Y$ , instead of an unknot we tie a copy of the knot  $K$ , with framing  $n$ . In other words, instead of plumbing sphere bundles, we plumb the trace of  $n$ -surgery along  $K$  using the co-core of the 2-handle.

Whenever  $S_{n_j^2}^3(K_j)$  bounds a rational homology ball for each  $j = 1, \dots, v$ , so does  $Y$  ([1]; see also [2]). We claim that if each  $n_j$  is odd,  $n_j$  and  $n_k$  are pairwise coprime for each  $j \neq k$ , and  $a \not\equiv v \pmod{2}$ , then  $H_1(Y)$  is cyclic. From now on, we will make these assumptions on  $n_j$  and  $a$ .

In order to prove the claim, let us first modify the diagram by blowing down the  $-1$ -vertex and one of the two  $-2$ -vertices, obtaining the following diagram:



Note that the double edge between the two top-most vertices is not to be intended in the plumbing diagram sense, but rather signifies a double intersection between the corresponding attaching circles; in particular, it *does not* increase  $b_1$ . Let  $P$  be the 4-manifold associated to the diagram above, with  $\partial P = Y$ .

Notice that  $P$  is a 2-handlebody, i.e. it is obtained from  $B^4$  by attaching only 2-handles, and hence  $H_1(P; R) = 0$  and  $H_2(P; R) = H_2(P) \otimes R$ , and  $H_2(P, Y; R) = H_2(P, Y) \otimes R$  for each ring  $R$ , and that the latter are both free over  $R$  of rank  $v + 2$ .

We now set out to compute  $H_1(Y)$  as the quotient of  $\mathbb{Z}^{v+2}$  by the image of the intersection matrix of the link coming from the diagram above. The matrix is

$$Q = \begin{pmatrix} a+2 & 2 & 1 & \cdots & 1 \\ 2 & 0 & 0 & \cdots & 0 \\ 1 & 0 & n_1^2 & & \\ \vdots & \vdots & & \ddots & \\ 1 & 0 & & & n_v^2 \end{pmatrix}.$$

By expanding along the second row, we easily see that  $\det Q = (\prod_{j=0}^v n_j)^2$ , where we let  $n_0 = 2$  for convenience.

In order to see that  $H_1(Y)$  is cyclic, since  $\det Q = \prod_{j=0}^v |\mathbb{Z}/n_j^2\mathbb{Z}|$ , it is enough to check that, for each  $j$ ,  $H_1(Y; \mathbb{Z}/n_j^2\mathbb{Z}) \cong \mathbb{Z}/n_j^2\mathbb{Z}$ . Let  $R = \mathbb{Z}/n_j^2\mathbb{Z}$ .

The long exact sequence for the pair  $(P, Y)$  yields:

$$\begin{array}{ccccccc} H_2(P; R) & \xrightarrow{\alpha} & H_2(P, Y; R) & \longrightarrow & H_1(Y; R) & \longrightarrow & 0 \\ \downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \\ R^{v+2} & \xrightarrow{Q_R} & R^{v+2} & \longrightarrow & \text{coker } Q_R & \longrightarrow & 0 \end{array}$$

where  $Q_R$  is the reduction of  $Q$  modulo  $n_j^2$ . It is enough to show that the quotient is cyclic. This is elementary from the matrix  $Q$ , since  $n_k$  is now invertible in  $R$  for every  $k \neq j$ . When  $j = 0$ , it is helpful (but not necessary) to use the fact that  $n_k^2 \equiv 1 \pmod{4}$  for each  $k > 0$ ; one then reduces to the case of the matrix  $\begin{pmatrix} a+2-v & 2 \\ 2 & 0 \end{pmatrix}$ , which is well-known to have cyclic cokernel precisely when  $a+2-v$  is odd.

Observe that, after doing a dot-zero surgery on the 0-framed unknot, the diagram above also exhibits a rational homology cobordism  $W$  from  $Y' := \#_{j=1}^v S_{n_j^2}^3(K_j)$  to  $Y$ . Moreover, it is easy to check that the inclusion induces an injection  $\iota'_* : H_1(Y') \rightarrow H_1(W)$ , hence every character  $\varphi'$  of  $Y'$  extends to  $W$ , and we can further restrict it to  $Y$ . Let  $\iota_* : H_1(Y) \rightarrow H_1(W)$  be the map induced by the inclusion. Additionally, since  $\text{coker } \iota'_* = \text{coker } \iota_* = \mathbb{Z}/2\mathbb{Z}$  and since  $|H_1(Y')|$  is odd, the order of the induced character on  $Y$  is either the order of  $\varphi'$  or twice as large.

We are going to look at a character  $\varphi : H_1(Y) \rightarrow \mathbb{C}^*$  induced as above from the character  $\varphi'$  on  $Y'$  that sends the meridian of  $K_j$  to  $\exp(2\pi i/n_j^2)$  for each  $j$ .

By additivity of the Casson–Gordon signature defects [11],

$$\sigma(Y', \varphi') = \sum_{j=1}^n \sigma(S_{n_j^2}^3(K_j), \varphi_j) = \sum_{j=1}^n 1 - \sigma_{K_j}(e^{2\pi i/n_j}) + 2(n_j - 1).$$

Let us now assume that  $K_j = T_{F_{p_j}^2, F_{p_j+2}^2}$ ,  $n_j = F_{p_j} F_{p_j+2}$ , where the sequence  $p_j$  is defined recursively by  $p_1 = 5$ ,  $p_{j+1} = 6 \prod_{k \leq j} (p_k^2 + 2p_k) - 1$ .

Since  $\gcd(F_a, F_b) = F_{\gcd(a,b)}$ , and since  $F_3 = 2$ , we have that  $n_j$  is odd for each  $j$ . Moreover, by construction,

$$p_j \equiv 1 \pmod{2}, \quad p_j \equiv -1 \pmod{p_k}, \quad p_j \equiv -1 \pmod{p_k + 2},$$

and hence both  $p_j$  and  $p_j + 2$  are odd and coprime with  $p_k$  for each  $k$ ; thus,  $n_j$  and  $n_k$  are coprime, too.

It also follows from work of Orevkov ([18]; see also [9, 5]) that  $S_{n_j^2}^3(K_j)$  bounds a rational homology ball, as seen in the previous example, too, and that we can choose these balls to be constructed using only 1- and 2-handles. Since  $W$  is a rational homology cobordism constructed with one 1-handle and one 2-handle,  $Y$  has the same property.

In fact, the rational homology balls of the previous example used two 1-handles, and  $W$  uses only one 1-handle, so  $Y$  bounds a rational homology ball constructed with  $2v + 1$  1-handles and  $2v + 1$  2-handles.

The computation in the previous example, together with additivity, shows that  $\sigma(Y', \varphi') = 2v$ .

Since  $\varphi'$  extends to the cobordism  $W$ , we can glue  $W$  to any 4-manifold  $Z'$  to which  $\varphi'$  extends (rationally), and use the resulting 4-manifold  $Z = W \cup Z'$  to compute the signature defect  $\sigma(Y, \varphi)$ .

Since  $W$  is a rational homology cobordism, the ordinary signature does not change; that is,  $\sigma(Z) = \sigma(Z')$ . The twisted signature  $\sigma^\psi(Z)$  is also controlled by  $\sigma^{\psi'}(Z')$ : indeed, since  $W$  contains a single 2-handle,  $|\sigma^\psi(Z) - \sigma^{\psi'}(Z')| \leq 1$  by Novikov additivity.

It follows that  $\sigma(Y, \varphi) \geq \sigma(Y', \varphi') - 1 = 2v - 1$ , and therefore any rational homology ball filling  $Y$ , that is built only using 1- and 2-handles, has at least  $2v - 1$  1-handles, by Corollary 2.2.

To conclude, we argue that  $Y$  is irreducible. Indeed, we can replace each of the nodes labelled with  $[K_j, n_j^2]$  above with a negative definite plumbing tree, and, using Neumann's criterion [17], one can check that the plumbing is in normal form, and its boundary is irreducible.

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